

ON μ -CONFORMAL HOMEOMORPHISMS AND BOUNDARY CORRESPONDENCE

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ABSTRACT. We study the boundary correspondence under μ -homeomorphisms f of the open upper half-plane onto itself. Sufficient conditions are given for f to admit a homeomorphic extension to the closed half-plane with prescribed boundary regularity. The proofs are based on the modulus estimates for semiannuli in terms of directional dilations of f which might be of independent interest.

1. INTRODUCTION AND MAIN THEOREMS

Let f be a quasiconformal homeomorphism of the open upper half-plane \mathbb{H} onto itself. A. Beurling and L. Ahlfors [5] have shown that f admits a homeomorphic extension to the closed half-plane and that its boundary map can be characterized as a quasisymmetric function on \mathbb{R} if it fixes the point at infinity. Moreover, they remark that the boundary map need not be locally absolutely continuous on \mathbb{R} . In [11], L. Carleson proved that the condition

$$\int_0^1 \frac{\eta(s)}{s} dt < \infty, \quad \eta(s) = \operatorname{ess\,sup}_{0 < \operatorname{Im} z < s} |\mu_f(z)|, \quad \mu_f = \frac{f_{\bar{z}}}{f_z},$$

implies that the boundary correspondence $t \mapsto f(t)$ has a continuous derivative $f'(t)$ for a quasiconformal self-mapping f of \mathbb{H} with $f(\infty) = \infty$.

M. Brakalova and J. Jenkins [10] have extended Carleson's theorem to a class of μ -conformal homeomorphisms. Here and hereafter, a *Beltrami coefficient* μ on a domain Ω will mean a complex-valued measurable function such that $|\mu| < 1$ a.e. in Ω and a homeomorphism $f : \Omega \rightarrow \Omega'$ will be called μ -conformal if f belongs to the Sobolev space $W_{\operatorname{loc}}^{1,1}(\Omega)$ and satisfies the Beltrami equation

$$f_{\bar{z}} = \mu f_z \quad \text{a.e. in } \Omega.$$

When $\|\mu\|_{\infty} < 1$ the function f is called *quasiconformal*.

For the present and later use, we set

$$A(z_0; r, R) = \{z \in \mathbb{C}; r \leq |z - z_0| \leq R\}$$

for $z_0 \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $0 \leq r < R \leq +\infty$. Here, $A(\infty; r, R)$ is defined to be $A(0; 1/R, 1/r)$.

Brakalova and Jenkins [10] proved the following.

Theorem A (Brakalova-Jenkins [10]). *Let f be a μ -conformal self-homeomorphism of the upper half-plane \mathbb{H} for a Beltrami coefficient μ . Suppose that $f(z) \rightarrow \infty$ if and only if*

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$z \rightarrow \infty$ in \mathbb{H} and that

$$\iint_{A(t,r,R) \cap \mathbb{H}} \frac{|\mu(z)|^2 + |\operatorname{Re} \frac{\bar{z}-t}{z-t} \mu(z)|}{1 - |\mu(z)|^2} \frac{dxdy}{|z-t|^2}, \quad z = x + iy,$$

converges as $r \rightarrow 0+$ for every $t \in \mathbb{R}$ and some $R = R(t) > 0$. Then f extends to a homeomorphism of $\overline{\mathbb{H}}$ in such a way that the boundary function $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable everywhere and

$$f'(t) = \lim_{z \rightarrow t \text{ in } \mathbb{H}} \frac{f(z) - f(t)}{z - t} > 0, \quad t \in \mathbb{R}.$$

Moreover if the above convergence is locally uniform for $t \in \mathbb{R}$, then f' is continuous.

In their theorem, continuity of the function at ∞ is assumed. As we will see later (Example 3.5), this does not follow from the other assumptions. We slightly refine their result and state it as a local version.

Theorem 1.1. *Let μ be a Beltrami coefficient on \mathbb{H} which satisfies the following two conditions for $t \in I$, where I is an open interval in \mathbb{R} :*

- (1) $\iint_{A(t,r_1,r_2) \cap \mathbb{H}} \frac{|\mu(z)|^2}{1 - |\mu(z)|^2} \frac{dxdy}{|z-t|^2} \rightarrow 0$ as $r_1, r_2 \rightarrow 0$, and
- (2) $\operatorname{Re} \iint_{A(t,r_1,r_2) \cap \mathbb{H}} \frac{\mu(z)}{(z-t)^2} \frac{dxdy}{1 - |\mu(z)|^2} \rightarrow 0$ as $r_1, r_2 \rightarrow 0$.

Suppose that there exists a μ -conformal self-homeomorphism f of \mathbb{H} . Then it extends to a homeomorphism of $\mathbb{H} \cup I$ into $\overline{\mathbb{H}}$. Furthermore, if $f(I) \subset \mathbb{R}$, then the boundary function $f : I \rightarrow \mathbb{R}$ is differentiable and

$$f'(t) = \lim_{z \rightarrow t \text{ in } \mathbb{H}} \left| \frac{f(z) - f(t)}{z - t} \right| > 0$$

holds for each $t \in I$. Moreover, if the convergence in (1) and (2) is locally uniform for $t \in I$, then the derivative f' is continuous on I .

A proof of this theorem will be given in the final section as well as those of the other results in the present section. Though the proof of Theorem 1.1 can be done in the same way as in [10], we will supply a detailed account for the continuous extension of the mapping f since it is largely omitted in [10]. To this end, in Section 3, we give an explicit estimate for the modulus of continuity of the boundary mapping at a given point. As preparations, we introduce the notion of semiannulus and give concrete estimates of the modulus of a semiannulus and related quantities in Section 2. We believe that these estimates and methods will be useful in various other problems. Indeed, as applications of the estimates of modulus of continuity, we propose a couple of related results in the rest of the present section.

We recall that a function f is locally Lipschitz continuous on a subset X of \mathbb{C} if for every compact subset E of X , there exists a constant $C = C(E)$ such that the inequality $|f(z) - f(z_0)| \leq C|z - z_0|$ holds for $z_0, z \in E$. The following result gives us a sufficient condition for a μ -conformal homeomorphism $f : \mathbb{H} \rightarrow \mathbb{H}$ to admit an extension to $\overline{\mathbb{H}}$

whose boundary correspondence is locally Lipschitz continuous. This can be regarded as a boundary version of Theorem 3.13 in [16].

Theorem 1.2. *Let $f : \mathbb{H} \rightarrow \mathbb{H}$ be a μ -conformal homeomorphism for a Beltrami coefficient μ on \mathbb{H} and let I be an open interval in \mathbb{R} . If there are positive constants R and M such that*

$$(1.3) \quad \iint_{A(t;r,R) \cap \mathbb{H}} \frac{|\mu(z)|^2 - \operatorname{Re} \frac{\bar{z}-t}{z-t} \mu(z)}{1 - |\mu(z)|^2} \frac{dxdy}{|z-t|^2} \leq M$$

for every $t \in I$ and $r \in (0, R)$, then f is extended to a homeomorphism of $\mathbb{H} \cup I$ into the closed upper half-plane. If moreover $f(I) \subset \mathbb{R}$, the boundary function $f : I \rightarrow \mathbb{R}$ is locally Lipschitz continuous on I .

We will say that a function f is *locally weak Hölder continuous* with exponent $\alpha > 0$ in a subset X of \mathbb{C} if for every compact subset E of X and every $0 < \alpha' < \alpha$ there is a constant $C = C(\alpha', E) > 0$ such that

$$|f(z) - f(z_0)| \leq C|z - z_0|^{\alpha'}$$

holds whenever $z_0, z \in E$. In particular, f is called *locally weak Lipschitz continuous* when $\alpha = 1$. We have now the following result, a prototype of which is Theorem 4.5 in [16].

Theorem 1.4. *Let $f : \mathbb{H} \rightarrow \mathbb{H}$ be a μ -conformal homeomorphism for a Beltrami coefficient μ . For a given open interval I in \mathbb{R} and a number $0 < \alpha \leq 1$, we suppose that*

$$(1.5) \quad \limsup_{r \rightarrow 0+} \frac{4}{\pi r^2} \iint_{A(t;0,r) \cap \mathbb{H}} \frac{|\mu(z)|^2 - \operatorname{Re} \frac{\bar{z}-t}{z-t} \mu(z)}{1 - |\mu(z)|^2} dxdy \leq \frac{1}{\alpha} - 1,$$

where the convergence is locally uniform for $t \in I$. Then f is extended to a homeomorphism of $\mathbb{H} \cup I$ into the closed upper half-plane $\overline{\mathbb{H}}$ and, if $f(I) \subset \mathbb{R}$ in addition, the boundary correspondence $t \mapsto f(t)$ is locally weak Hölder continuous with exponent α on I .

In order to obtain a result of the same type as Theorem A, we have to impose another condition on μ .

Lemma 1.6. *Let f be a μ -conformal self-homeomorphism of \mathbb{H} for a Beltrami coefficient μ on \mathbb{H} . If*

$$\lim_{R \rightarrow +\infty} \frac{1}{(\log R)^2} \iint_{A(0;r_0,R) \cap \mathbb{H}} \frac{|\mu(z)|^2 - \operatorname{Re} \frac{\bar{z}}{z} \mu(z)}{1 - |\mu(z)|^2} \frac{dxdy}{|z|^2} = 0$$

for some $r_0 > 0$, then f extends continuously to the point at infinity.

We will show the lemma in the last section. The extended map in the lemma does not necessarily satisfy $f(\infty) = \infty$. However, $g = M \circ f$ becomes a μ -conformal homeomorphism of \mathbb{H} with $g(\infty) = \infty$ for a Möbius transformation M such that $M(\mathbb{H}) = \mathbb{H}$ and $M(f(\infty)) = \infty$.

We may add the condition in this lemma as well as $I = \mathbb{R}$ to the assumptions in the above theorems to guarantee a homeomorphic extension of f to the closed upper half-plane.

We also have results analogous to the above theorems for the case of the unit disk. As a sample, we give a variant of Theorem 1.1. In the following, let $T(\zeta; r, R) = \{z \in \mathbb{D} : r < |\frac{z-\zeta}{z+\zeta}| < R\}$ for $\zeta \in \partial\mathbb{D}$ and $0 < r < R < +\infty$.

Theorem 1.7. *Let μ be a Beltrami coefficient on the unit disk \mathbb{D} . Assume the following two conditions for $\zeta \in \partial\mathbb{D}$:*

- (i) $\iint_{T(\zeta; r_1, r_2)} \frac{|\mu(z)|^2}{1 - |\mu(z)|^2} \frac{dxdy}{|z^2 - \zeta^2|^2} \rightarrow 0$ as $r_1, r_2 \rightarrow 0$, and
- (ii) $\operatorname{Re} \iint_{T(\zeta; r_1, r_2)} \frac{\zeta^2 \mu(z)}{(z^2 - \zeta^2)^2} \frac{dxdy}{1 - |\mu(z)|^2} \rightarrow 0$ as $r_1, r_2 \rightarrow 0$.

Suppose that there exists a μ -conformal self-homeomorphism f of \mathbb{D} . Then it extends to a self-homeomorphism of $\overline{\mathbb{D}}$ and the boundary map $f : \partial\mathbb{D} \rightarrow \partial\mathbb{D}$ is differentiable and

$$\left| \frac{d}{d\theta} f(e^{i\theta}) \right| = \lim_{z \rightarrow e^{i\theta} \text{ in } \mathbb{D}} \left| \frac{f(z) - f(e^{i\theta})}{z - e^{i\theta}} \right| > 0$$

holds for each $\theta \in \mathbb{R}$. Moreover, if the convergence in (i) and (ii) is uniform for $\zeta \in \partial\mathbb{D}$, then the derivative of f along the unit circle is continuous.

2. MODULUS OF SEMIANNULUS

In the study of regularity for quasiconformal mappings (or more general homeomorphisms), the notion of ring domain (annulus) plays an important role. In order to study the boundary regularity for homeomorphisms, we need its counterpart for the boundary. In this section, we introduce the notion of semiannulus and present its basic properties. See also a survey article [21] by the third author for expository accounts.

In what follows, we will consider subsets of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Therefore, it is sometimes convenient to introduce the spherical (chordal) distance

$$d^\#(z, w) = \frac{|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}.$$

The spherical diameter of a set A will be denoted by $\operatorname{diam}^\# A$.

A subset S of $\widehat{\mathbb{C}}$ is called a *semiannulus* if it is homeomorphic to

$$T_R = \{z \in \mathbb{C} : 1 \leq |z| \leq R, \operatorname{Im} z > 0\}$$

for some $R \in (1, +\infty)$. The two simple open arcs in the boundary of S which correspond to $\{|z| = 1, \operatorname{Im} z > 0\}$ and $\{|z| = R, \operatorname{Im} z > 0\}$ are called the *sides* of S . The complementary boundary components of S are called the *ends* of S .

A semiannulus S in a plane domain D is said to be *properly embedded* in D if $S \cap K$ is compact whenever K is a compact subset of D .

A semiannulus S is said to be *conformally equivalent* to another semiannulus S' if there is a homeomorphism $f : S \rightarrow S'$ which is conformal in $\operatorname{Int} S$. We define the modulus of S somewhat artificially as follows: Set $\operatorname{mod} S = \log R$ when S is conformally equivalent to T_R . If there is no such an $R > 1$, we set $\operatorname{mod} S = 0$.

One can also define $\operatorname{mod} S$ in an intrinsic way. For that, we use the extremal length $\lambda(\Gamma)$ of a curve family Γ (see [1] for the definition and its fundamental properties).

Let Γ_S be the collection of open arcs in S dividing the two sides of S and Γ'_S be that of closed arcs in S joining the two sides of S . Here, a curve γ in S is called dividing if the sides of S are contained in different connected components of $S \setminus \gamma$. Then we have the following.

Lemma 2.1. *Let S be a semiannulus in $\widehat{\mathbb{C}}$. Then*

$$\text{mod } S = \frac{\pi}{\lambda(\Gamma_S)} = \pi \lambda(\Gamma'_S).$$

Furthermore, $\text{mod } S = 0$ if and only if there exists a sequence of simple closed arcs γ_n , $n = 1, 2, 3, \dots$, in S joining the two sides of S such that $\text{diam}^\# \gamma_n \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Let $h : \text{Int } S \rightarrow \mathbb{D}$ be a Riemann mapping function. By Carathéodory's theory of prime ends (see [19] for details), the prime ends of $\text{Int } S$ correspond to the boundary points of \mathbb{D} through the function h in a one-to-one fashion. The sides of S are open simple arcs in ∂S , and therefore, these correspond to disjoint open circular arcs, say, O_1 and O_2 in $\partial \mathbb{D}$ and h extends to a homeomorphism of S onto the semiannulus $S' = \mathbb{D} \cup (O_1 \cup O_2)$. Let C_1 and C_2 be the connected components of $\partial \mathbb{D} \setminus (O_1 \cup O_2)$. Clearly, $h(\Gamma_S) = \Gamma_{S'}$ is the collection of open arcs in S' dividing the sides O_1 and O_2 . On the other hand, $h(\Gamma'_S) = \Gamma'_{S'}$ is the collection of closed arcs in S' joining O_1 and O_2 . When we regard S' as a quadrilateral, the curve families $\Gamma_{S'}$ and $\Gamma'_{S'}$ are conjugate to each other and therefore $\lambda(\Gamma_{S'}) = 1/\lambda(\Gamma'_{S'})$ (see [1]). Since the extremal length is conformally invariant, we have $\lambda(\Gamma_S) = 1/\lambda(\Gamma'_S)$.

Obviously, $\lambda(\Gamma_S) = \lambda(\Gamma'_S) = +\infty$ if and only if either C_1 or C_2 reduces to one point, equivalently, one of the two ends of S is a prime end. By the definition of prime ends, this means existence of a sequence of simple closed arcs γ_n (called a null-chain) in S with the following properties: γ_n joins the two sides of S , γ_n separates γ_{n-1} from γ_{n+1} in S , and $\text{diam}^\# \gamma_n \rightarrow 0$.

Finally, we prove $\text{mod } S = \pi/\lambda(\Gamma_S)$. This is certainly true when $\text{mod } S = 0$ by the above observation. Thus we can assume that $\text{mod } S > 0$. Then there exists a number $0 < a < +\infty$ such that $S' = \mathbb{D} \cup O_1 \cup O_2$ is conformally equivalent to the rectangle $S_0 = \{x + iy : 0 \leq x \leq a, 0 < y < \pi\}$. Furthermore, the function e^z maps S_0 onto T_{e^a} conformally inside. Therefore, we now have $\text{mod } S = \log e^a = a$ by definition. On the other hand, as is well known (cf. [1]), we have $\lambda(\Gamma_{S_0}) = \pi/a$. Hence, $\text{mod } S = a = \pi/\lambda(\Gamma_{S_0}) = \pi/\lambda(\Gamma_S)$. \square

In particular, if the two sides of S have a positive spherical distance, $\text{mod } S > 0$. If S is properly embedded in a good enough domain, then this sort of condition is indeed characterizing positivity of the modulus. For instance, we can show the following.

Corollary 2.2. *Let S be a semiannulus properly embedded in the unit disk \mathbb{D} and let U_1 and U_2 be the connected components of $\mathbb{D} \setminus S$. Then $\text{mod } S > 0$ if and only if the Euclidean distance between U_1 and U_2 is positive.*

Proof. First note that $\text{dist}(U_1, U_2) = \text{dist}(\sigma_1, \sigma_2)$, where “dist” stands for the Euclidean distance and σ_1, σ_2 are the sides of S . Since the spherical distance is comparable with the Euclidean distance in \mathbb{D} , the conclusion follows from the above lemma. \square

The following result will constitute the basis of boundary estimates for a disk homeomorphism.

Theorem 2.3. *Let S be a semiannulus properly embedded in \mathbb{D} and U_1 and U_2 be the two connected components of $\mathbb{D} \setminus S$. Then*

$$\min\{\text{diam } U_1, \text{diam } U_2\} \leq C \exp(-\tfrac{1}{2} \bmod S),$$

where $C = 4e^{\pi/2}$.

For the proof of the theorem, we have to prepare a couple of lemmas. Let S be a semiannulus properly embedded in the unit disk \mathbb{D} . If $\bmod S$ is positive, by Lemma 2.1, the Euclidean distance between the two sides of S is positive. Therefore, the interior of the set $\overline{S} \cup \{1/\bar{z} : z \in S\}$ becomes a ring domain and will be denoted by \hat{S} . The modulus of a ring domain B is defined to be $\log R$ if B is conformally equivalent to the standard annulus $1 < |z| < R$. We now have the following by a symmetry principle.

Lemma 2.4. *Let S be a semiannulus properly embedded in the unit disk. Then $\bmod S = \bmod \hat{S}$ whenever $\bmod S > 0$.*

Proof. Let $R = \exp(\bmod S) > 1$. By definition, there is a homeomorphism $f : S \rightarrow T_R$ which is conformal on $\text{Int } S$. By the Schwarz reflection principle, $f|_{\text{Int } S}$ can be continued analytically to a conformal homeomorphism $f : \hat{S} \rightarrow \{w \in \mathbb{C} : 1 < |w| < R\}$. Therefore, $\bmod \hat{S} = \log R = \bmod S$. \square

We recall a sort of separation lemma for an annulus, which can date back to Teichmüller's work in 1930's. The following sharp form is due to Avkhadiev and Wirths [3] (see also [4, Theorem 3.17] or [21]).

Lemma 2.5 (Avkhadiev-Wirths). *Let B be a ring domain in \mathbb{C} with $\bmod B > \pi$ which separates a given point $z_0 \in \mathbb{C}$ from ∞ . Then there is a ring domain A contained in B of the form $\{z : r < |z - z_0| < R\}$ such that $\bmod A = \log \frac{R}{r} \geq \bmod B - \pi$. The constant π is sharp.*

A subset S_0 of a semiannulus S is called a *subsemiannulus* of S if S_0 is a semiannulus satisfying $\Gamma_{S_0} \subset \Gamma_S$. Since $\lambda(\Gamma_{S_0}) \geq \lambda(\Gamma_S)$, we have $\bmod S_0 \leq \bmod S$ by Lemma 2.1.

For $\zeta \in \partial\mathbb{D}$ and $0 < r_1 < r_2 < +\infty$, we set

$$(2.6) \quad T(\zeta; r_1, r_2) = \{z \in \mathbb{D} : r_1 \leq |\frac{z-\zeta}{z+\zeta}| \leq r_2\}.$$

Note that $\hat{T}(\zeta; r_1, r_2) = \{z \in \hat{\mathbb{C}} : r_1 < |\frac{z-\zeta}{z+\zeta}| < r_2\}$ and

$$\bmod T(\zeta; r_1, r_2) = \bmod \hat{T}(\zeta; r_1, r_2) = \log \frac{r_2}{r_1}.$$

The following result is a hyperbolic analog of Lemma 2.7 in [17].

Lemma 2.7. *Let T be a semiannulus properly embedded in \mathbb{D} whose sides are circular arcs perpendicular to $\partial\mathbb{D}$ and let V_1 and V_2 be the connected components of $\mathbb{D} \setminus T$. Then*

$$\min\{\text{diam } V_1, \text{diam } V_2\} \leq \frac{2}{\cosh(\frac{1}{2} \bmod T)}.$$

Equality holds if and only if T is of the form $T(\zeta; r, 1/r)$ for some $\zeta \in \partial\mathbb{D}$ and $0 < r < 1$.

Proof. We denote by d_Ω the hyperbolic distance on a hyperbolic domain Ω .

Let C_1 and C_2 be the sides of T and let δ denote the hyperbolic distance between C_1 and C_2 in \mathbb{D} . There is a unique hyperbolic line C in \mathbb{D} such that the hyperbolic length of $C \cap T$ is δ , in other words, $C \cap T$ is the hyperbolic geodesic joining C_1 and C_2 .

Let ζ_1 and ζ_2 be the endpoints of C with $\zeta_j \in \partial V_j \cap \partial\mathbb{D}$, $j = 1, 2$. We now define a conformal homeomorphism L of \mathbb{D} onto the right half-plane H by

$$L(z) = \frac{\zeta_2 + z}{\zeta_2 - z} - \frac{\zeta_2 + \zeta_1}{\zeta_2 - \zeta_1}.$$

Then $L(C)$ is the half-line $(0, +\infty)$ and $L(C_1)$ and $L(C_2)$ are concentric circular arcs centered at the origin. We denote by r_1 and r_2 the radii of those circles. Then the hyperbolic distance between V_1 and V_2 in \mathbb{D} can be computed by

$$\delta = d_\mathbb{D}(V_1, V_2) = d_H(L(V_1), L(V_2)) = \int_{r_1}^{r_2} \frac{dx}{2x} = \frac{1}{2} \log \frac{r_2}{r_1} = \frac{1}{2} \bmod T.$$

Thus the problem now reduces to finding a configuration of two hyperbolic half-planes V_1 and V_2 with a fixed hyperbolic distance such that the minimum of their Euclidean diameters is maximal (namely, the worst case). Such a configuration is attained obviously by the situation that $V_2 = -V_1$. In this case, C becomes a line segment passing through the origin. By a suitable rotation, we may assume that $\zeta_1 = 1, \zeta_2 = -1$. Let $a > 0$ be the number determined by $V_1 \cap \mathbb{R} = (a, 1)$. Since 0 is the midpoint of the geodesic $[-a, a]$ joining V_1 and V_2 , we have $\delta/2 = d_\mathbb{D}(0, a) = \operatorname{arctanh} a$ and $a = \tanh(\delta/2)$. The disk automorphism (hyperbolic isometry) $g(z) = (z + a)/(1 + az)$ maps the hyperbolic half-plane $\{z \in \mathbb{D} : \operatorname{Re} z > 0\}$ onto V_1 . Therefore, we see that $g(i)$ and $g(-i)$ are the endpoints of $C_1 = \partial V_1 \cap \mathbb{D}$ and thus $\operatorname{diam} V_1 = |g(i) - g(-i)| = 2(1 - a^2)/(1 + a^2)$. Finally, we get the estimate in this case

$$\operatorname{diam} V_j = 2 \frac{1 - \tanh(\delta/2)^2}{1 + \tanh(\delta/2)^2} = \frac{2}{\cosh \delta}.$$

Since $\delta = \frac{1}{2} \bmod T$, the estimate is now shown. The equality case is obvious from the above argument. \square

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. When $\bmod S \leq \pi$, the assertion trivially holds. We now suppose that $\bmod S > \pi$. Take points $\zeta_j \in \overline{U_j} \cap \partial\mathbb{D}$ ($j = 1, 2$) and let $L(z) = (z + \zeta_2)/(z - \zeta_2) - (\zeta_1 + \zeta_2)/(\zeta_1 - \zeta_2)$. Then, by Lemma 2.5, $L(\hat{S})$ contains a ring domain A of the form $\{w : r_1 < |w| < r_2\}$ with $\bmod A = \log \frac{r_2}{r_1} \geq \bmod \hat{S} - \pi = \bmod S - \pi$. Set $T = L^{-1}(\overline{A}) \cap \mathbb{D}$ and let V_1, V_2 be the two components of $\mathbb{D} \setminus T$ so that $U_j \subset V_j$ ($j = 1, 2$). By Lemma 2.7,

we have

$$\begin{aligned}
\min\{\text{diam } U_1, \text{diam } U_2\} &\leq \min\{\text{diam } V_1, \text{diam } V_2\} \\
&\leq \frac{2}{\cosh(\frac{1}{2} \bmod T)} \\
&< 4 \exp(-\frac{1}{2} \bmod T) \\
&< 4 \exp(-\frac{1}{2} \bmod S + \pi/2).
\end{aligned}$$

□

We also have a variant of Theorem 2.3 for the case of half-planes. It is an important point that we do not lose one half of the modulus in the estimate with the expense of a condition for the modulus. Note that a prototype can be found at [16, Lemma 2.8] and [17, Lemma 2.8].

Theorem 2.8. *Let S be a semiannulus properly embedded in \mathbb{H} and U_1 and U_2 be the two connected components of $\mathbb{H} \setminus S$. If $\bmod S > \pi$ and if U_2 is unbounded, then for any point $t_0 \in \overline{U_1} \cap \mathbb{R}$, the inequality*

$$\sup_{z \in U_1} |z - t_0| \leq C_1 \text{dist}(t_0, U_2) \exp(-\bmod S)$$

holds, where $C_1 = e^\pi$.

Proof. Let \hat{S} be the ring domain obtained by reflecting $\text{Int } S$ in \mathbb{R} . Then $\bmod \hat{S} = \bmod S > \pi$ by the same argument as in Lemma 2.4. Since \hat{S} separates t_0 from ∞ , Lemma 2.5 guarantees existence of numbers $0 < r < R < +\infty$ such that $A = A(t_0; r, R) \subset \hat{S}$ and $\bmod A \geq \bmod S - \pi$. Since $\text{dist}(t_0, U_2) \geq R$, we now have

$$\sup_{z \in U_1} |z - t_0| \leq r = R \exp(-\bmod A(t_0; r, R)) \leq \text{dist}(t_0, U_2) \exp(\pi - \bmod S).$$

□

3. BOUNDARY CONTINUITY OF A HOMEOMORPHISM

The following simple example shows that a self-homeomorphism of the unit disk \mathbb{D} (sometimes called a disk homeomorphism) does not necessarily have a continuous extension to the boundary:

$$f(re^{i\theta}) = r \exp i(\theta - \log(1 - r)).$$

Here is a criterion of continuous extendibility of a disk homeomorphism to a boundary point. We recall that $T(\zeta; r, R)$ is defined in (2.6).

Proposition 3.1. *Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be a homeomorphism and let $\zeta \in \partial\mathbb{D}$. The mapping f extends continuously to ζ if*

$$\lim_{r \rightarrow 0+} \bmod f(T(\zeta; r, R)) = +\infty$$

for some $R > 0$.

Proof. Let U_r be the connected component of $\mathbb{D} \setminus T(\zeta; r, R)$ with $\zeta \in \overline{U_r}$ for $0 < r < R$ and let V_R be the other one, which does not depend on r . Then the family of the sets U_r , $0 < r < R$, constitutes a fundamental system of neighbourhoods of ζ . Theorem 2.3 now yields

$$\min\{\text{diam } f(U_r), \text{diam } f(V_R)\} \leq C \exp(-\tfrac{1}{2} \bmod f(T(\zeta; r, R))).$$

By assumption, the last term tends to 0 as $r \rightarrow 0+$. Since $\text{diam } f(V_R)$ is a fixed number, this implies that $\text{diam } \overline{f(U_r)} \rightarrow 0$ as $r \rightarrow 0$. Therefore, the intersection $\bigcap_{0 < r < R} \overline{f(U_r)}$ consists of a single point. We can now assign this point as the extended value of f at ζ so that f has a continuous extension to ζ . \square

We remark that the converse is not true in the last proposition. Indeed, consider the homeomorphism $f : \mathbb{D} \rightarrow \mathbb{D}$ determined by $f(\bar{z}) = \overline{f(z)}$, $z \in \mathbb{D}$ and

$$f(re^{i\theta}) = r \exp i\pi(\theta/\pi)^{-\log(1-r)}, \quad 0 \leq \theta \leq \pi, \quad 0 < r < 1.$$

Then, by construction, f extends to 1 continuously by setting $f(1) = 1$. However, since $f(re^{i\theta}) \rightarrow 1$ as $r \rightarrow 1-$ for any fixed θ with $|\theta| < \pi$, the converse of the proposition does not hold (see the proof of the next theorem).

If the assumption of the last proposition is true for every point of a non-degenerate subinterval of $\partial\mathbb{D}$, then the converse holds. As an application of our previous observations, we show indeed the following theorem.

Theorem 3.2. *Let E be a subset of $\partial\mathbb{D}$ and $f : \mathbb{D} \rightarrow \mathbb{D}$ be a homeomorphism. Suppose that for every $\zeta \in E$,*

$$\lim_{r \rightarrow 0+} \bmod f(T(\zeta; r, R)) = +\infty$$

holds true for some number $R = R(\zeta) > 0$. Then f extends to a continuous injective mapping of $\mathbb{D} \cup E$ into $\overline{\mathbb{D}}$.

Since $\overline{\mathbb{D}}$ is a compact Hausdorff space, the inverse mapping of a continuous bijection of $\overline{\mathbb{D}}$ onto itself is also continuous. Therefore, as an immediate corollary, we have the following result of Brakalova [7]. Note that, earlier than it, Jixiu Chen, Zhiguo Chen and Chengqi He [12] proved a similar result in a special situation (see also the proof of Lemma 2.3 in [13]).

Corollary 3.3 (Brakalova [7]). *A homeomorphism $f : \mathbb{D} \rightarrow \mathbb{D}$ admits a homeomorphic extension to $\overline{\mathbb{D}}$ if and only if for each $\zeta \in \partial\mathbb{D}$, there is an $R = R(\zeta) > 0$ such that*

$$\lim_{r \rightarrow 0+} \bmod f(T(\zeta; r, R)) = +\infty.$$

Proof of Theorem 3.2. By Proposition 3.1, f can be extended continuously to every point in E . It is true even in the context of General Topology that the extended mapping $f : \mathbb{D} \cup E \rightarrow \overline{\mathbb{D}}$ is indeed continuous.

We next show that f is injective in E . First we observe that $f(E) \subset \partial\mathbb{D}$. Therefore, if we assume that the conclusion does not hold, then $f(\zeta_1)$ and $f(\zeta_2)$ are the same point, say, ω_0 , for some $\zeta_1, \zeta_2 \in E$ with $\zeta_1 \neq \zeta_2$. We may take $R_j = R(\zeta_j)$ so small that

$T(\zeta_1; r_1, R_1) \cap T(\zeta_2; r_2, R_2) = \emptyset$ for $0 < r_1 < R_1, 0 < r_2 < R_2$. Let U_1 and U_2 be the connected components of $\mathbb{D} \setminus T$ with $\zeta_1 \in \overline{U_1}$, where $T = T(\zeta_1; r_1, R_1)$

Take sequences $z_n, z'_n \in \mathbb{D}$, $n = 1, 2, \dots$, so that $z_n \rightarrow \zeta_1$ and $z'_n \rightarrow \zeta_2$ as $n \rightarrow \infty$. Since $z_n \in U_1$ and $z'_n \in U_2$ for a sufficiently large n , one has $\text{dist}(f(U_1), f(U_2)) \leq |f(z_n) - f(z'_n)|$. We now let $n \rightarrow \infty$ to obtain $\text{dist}(f(U_1), f(U_2)) = 0$, which implies $\text{mod } f(T) = 0$ by Corollary 2.2. This contradicts the condition $\text{mod } f(T(\zeta_1; r_1, R_1)) \rightarrow +\infty$ as $r_1 \rightarrow 0+$. \square

In the same way, we have a half-plane version of the last theorem.

Theorem 3.4. *A homeomorphism f of the upper half-plane \mathbb{H} admits a homeomorphic extension to $\overline{\mathbb{H}}$ if and only if for each $t \in \partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$,*

$$\lim_{r \rightarrow 0+} \text{mod } f(A(t; r, R) \cap \mathbb{H}) = +\infty$$

for some $R = R(t) > 0$.

We recall that $A(\infty; r, R)$ is defined as $A(0; 1/R, 1/r)$. We end the present section with an illustrating example.

Example 3.5. Firstly, consider the function $\tanh(\frac{\pi}{4}z)$, which maps the parallel strip $0 < \text{Im } z < 1$ conformally onto $\{w \in \mathbb{D} : \text{Im } w > 0\}$, the upper half of the unit disk. Observe that the line $\text{Im } z = 1$ is mapped by the function to the semicircle $|w| = 1, \text{Im } w > 0$. We now define a self-homeomorphism f of the upper half-plane \mathbb{H} by

$$f(x + iy) = \begin{cases} \tanh \frac{\pi}{4}(x + iy) & (0 < y \leq 1), \\ y \tanh \frac{\pi}{4}(x + i) & (1 < y). \end{cases}$$

Then $f : \mathbb{H} \rightarrow \mathbb{H}$ is a homeomorphism and conformal in $0 < \text{Im } z < 1$. By construction, f is not continuous at ∞ but satisfies the assumptions of Theorem A except for the continuity at ∞ . This example shows also that we cannot replace $\mathbb{R} \cup \{\infty\}$ by \mathbb{R} in Theorem 3.4.

4. PROOF OF MAIN THEOREMS

In this section, we will give proofs of our main theorems stated in Section 1. As we will see soon, we can show even slightly stronger assertions. We need to introduce some technical quantities.

For a Beltrami coefficient μ ,

$$K_\mu = \frac{1 + |\mu|}{1 - |\mu|}$$

is sometimes called the *pointwise maximal dilatation* of μ (or, of a μ -conformal homeomorphism f). The μ -conformal homeomorphism f is K -quasiconformal precisely when $K_\mu \leq K$ a.e. The pointwise maximal dilatation is useful to measure quasiconformality of a μ -conformal homeomorphism. However, it is occasionally necessary to look at not only

the absolute value but also the argument of μ . For a Beltrami coefficient μ on a domain Ω and a point $z_0 \in \mathbb{C}$ (not necessarily in the domain Ω), set

$$D_{\mu, z_0}(z) = \frac{\left|1 - \mu(z) \frac{\bar{z} - \bar{z}_0}{z - z_0}\right|^2}{1 - |\mu(z)|^2} = \frac{|1 - e^{-2i\theta} \mu(z)|^2}{1 - |\mu(z)|^2}$$

for $z \in \Omega$, where $\theta = \arg(z - z_0)$. This quantity is sometimes called the directional dilatation and it was introduced by Andreian Cazacu [2]. This notion was effectively used by Reich and Walczak [20], Lehto [18] and later by Brakalova and Jenkins [6], [10], Brakalova [8], [9], the first and third authors [16], Martio and the first author [15], and Martio, Vuorinen and the authors [17].

It is easy to verify the inequalities

$$\frac{1}{K_\mu(z)} \leq D_{\mu, z_0}(z) \leq K_\mu(z), \quad z \in \Omega.$$

For a Beltrami coefficient μ on the upper half-plane \mathbb{H} , we consider the quantity

$$Q_\mu(t; r, R) = \frac{1}{\pi \log(R/r)} \iint_{A(t; r, R) \cap \mathbb{H}} \frac{D_{\mu, t}(z)}{|z - t|^2} dx dy$$

for $t \in \mathbb{R}$, and $0 < r < R < +\infty$.

In terms of $Q_\mu(t; r, R)$ and $D_{\mu, t}$, we have distortion estimates for the modulus of a semiannulus under the μ -conformal homeomorphism. The following are variants of Proposition 2.4 and Corollary 2.13 in [16] (see also [15, Lemma 2.5] for (4.3)). We omit the proof because we can show them in the same way as in [16].

Lemma 4.1. *Let μ be a Beltrami coefficient on the upper half-plane \mathbb{H} and f be a μ -conformal homeomorphism of \mathbb{H} onto another domain. Then, for the semiannulus $T = A(t; r, R) \cap \mathbb{H}$*

$$(4.2) \quad \frac{1}{Q_\mu(t; r, R)} \leq \frac{\text{mod } f(T)}{\text{mod } T} \leq Q_{-\mu}(t; r, R)$$

and

$$(4.3) \quad -\frac{1}{\pi} \iint_T \frac{D_{-\mu, t}(z) - 1}{|z - t|^2} dx dy \leq \text{mod } T - \text{mod } f(T) \leq \frac{1}{\pi} \iint_T \frac{D_{\mu, t}(z) - 1}{|z - t|^2} dx dy.$$

By making use of the last lemma, we are now able to prove Lemma 1.6.

Proof of Lemma 1.6. We first note that

$$Q_\mu(0; r_0, R) - 1 = \frac{2}{\pi \log(R/r_0)} \iint_{A(0; r_0, R) \cap \mathbb{H}} \frac{|\mu(z)|^2 - \text{Re } \frac{\bar{z}}{z} \mu(z)}{1 - |\mu(z)|^2} \frac{dx dy}{|z|^2}.$$

Hence, the condition in the lemma means that $Q_\mu(0; r_0, R) = 1 + o(\log R) = o(\log R)$ as $R \rightarrow +\infty$. By (4.2), we have

$$\text{mod } f(A(0; r_0, R) \cap \mathbb{H}) \geq \frac{\log(R/r_0)}{Q_\mu(0; r_0, R)}.$$

The last term blows up when $R \rightarrow +\infty$. Thus we now apply Theorem 3.4 to obtain the desired conclusion. \square

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. First we observe the relations

$$\frac{D_{\mu,t}(z) + D_{-\mu,t}(z)}{2} - 1 = \frac{2|\mu(z)|^2}{1 - |\mu(z)|^2}$$

and

$$\frac{D_{\mu,t}(z) - D_{-\mu,t}(z)}{2} = \frac{-2|z - t|^2}{1 - |\mu(z)|^2} \operatorname{Re} \frac{\mu(z)}{(z - t)^2}.$$

Therefore, conditions (1) and (2) in Theorem 1.1 mean that the functions (4.4)

$$G(r) = \iint_{A(t;r,R_0) \cap \mathbb{H}} \frac{D_{\mu,t}(z) - 1}{|z - t|^2} dx dy \quad \text{and} \quad H(r) = \iint_{A(t;r,R_0) \cap \mathbb{H}} \frac{D_{-\mu,t}(z) - 1}{|z - t|^2} dx dy$$

both have finite limits as $r \rightarrow 0 +$. (We remark here that similar conditions appear in [9, Theorem 1.3] in connection with conformality condition at a point.) By (4.3), we obtain $\operatorname{mod} f(A(t;r,R_0) \cap \mathbb{H}) = \log \frac{1}{r} + O(1)$ when $r \rightarrow 0 +$. In particular, $\operatorname{mod} f(A(t;r,R_0) \cap \mathbb{H}) \rightarrow +\infty$ as $r \rightarrow 0 +$. Therefore, by Proposition 3.1, we see that f can be extended to a homeomorphism of $\mathbb{H} \cup I$ into $\overline{\mathbb{H}}$.

From now on, we assume that $f(I) \subset \mathbb{R}$. Again, by (4.3), we have

$$H(r_1) - H(r_2) \leq \operatorname{mod} (f(A(t;r_1,r_2) \cap \mathbb{H})) - \log \frac{r_2}{r_1} \leq G(r_2) - G(r_1).$$

Since $G(r)$ and $H(r)$ have finite limits as $r \rightarrow 0 +$,

$$(4.5) \quad \lim_{r_2 \rightarrow 0+} \sup_{r_1 \in (0,r_2)} \left(\operatorname{mod} (f(A(t;r_1,r_2) \cap \mathbb{H})) - \log \frac{r_2}{r_1} \right) = 0.$$

Therefore, by applying the argument of the proof of Lemma 4.1 in [6] to a nesting sequence of semiannuli, we can show that the non-zero finite limit

$$g(t) := \lim_{z \rightarrow t \text{ in } \mathbb{H}} \left| \frac{f(z) - f(t)}{z - t} \right|$$

exists for each $t \in I$. Since the limit is unrestricted as long as $\operatorname{Im} z > 0$, we can let $\operatorname{Im} z \rightarrow 0$ to obtain

$$g(t) = \lim_{x \rightarrow t \text{ in } \mathbb{R}} \left| \frac{f(x) - f(t)}{x - t} \right| = \lim_{x \rightarrow t \text{ in } \mathbb{R}} \frac{f(x) - f(t)}{x - t},$$

which is nothing but the derivative of $f(t)$.

When the convergence in (1) and (2) of the theorem is locally uniform for $t \in I$, convergence in (4.5) is also locally uniform for $t \in I$. We can show now continuity of $f'(t)$ by the argument same as in [10]. \square

The above proof was done along the same line as in [6] and [10]. Therefore, we omitted details if the same argument in those papers can be applied. We should, however, note that the regularity assumptions on μ -conformal homeomorphisms are not clearly stated in their papers. At least, their proof can be justified when f belongs to the Sobolev space $W_{\text{loc}}^{1,1}$, which is equivalent to that f is an ACL homeomorphism with locally integrable partial derivatives (see [14, §4.9.2] or [22, Theorem 2.1.4] for instance). There is still a

point where we should be careful. We may extend the above $f : \mathbb{H} \cup I \rightarrow \overline{\mathbb{H}}$ furthermore to the lower half-plane $\mathbb{H}^- = \{z : \operatorname{Im} z < 0\}$ by setting $f(z) = \overline{f(\bar{z})}$. Unlike the case of quasiconformal mappings, it is not clear that the extended f belongs to the Sobolev space $W_{\text{loc}}^{1,1}(\mathbb{H} \cup I \cup \mathbb{H}^-)$. We give a simple example below to show that this is not necessarily true in a general setting. Therefore, the results in [6] might not be applied directly under the current situations. However, as we suggested in the proof, the arguments in [6] work when we replace annuli by semiannuli in a suitable way.

Example 4.6. Define a function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ by $\phi(y) = y \sin(1/y)$ if $0 < |y| < 1/\pi$ and $\phi(y) = 0$ otherwise. Then the function

$$f(x + iy) = x + \phi(y) + iy, \quad x, y \in \mathbb{R},$$

is a self-homeomorphism of \mathbb{C} satisfying $\overline{f(z)} = f(\bar{z})$. Since the restriction of f to \mathbb{H} is a self-diffeomorphism, $f \in W_{\text{loc}}^{1,1}(\mathbb{H})$. However, the image of the segment $[x, x + i/\pi]$ is of infinite length for every $x \in \mathbb{R}$, and therefore, f is not ACL. In particular, f does not belong to $W_{\text{loc}}^{1,1}(\mathbb{C})$.

Remark 4.7. In order to obtain convergence of the argument of $(f(z) - f(t))/(z - t)$, we shall need the convergence of the integrals as in (4.4) over the sets $\{z : r \leq |z - t| \leq R_0, \theta_1 < \arg(z - t) < \theta_2\}$ for $0 < r < R_0, 0 \leq \theta_1 < \theta_2 \leq \pi$ (see [6, §5]). This condition is implied by the assumptions of Theorem A but not by those of Theorem 1.1.

Proof of Theorem 1.2. Take a fixed point $z_0 \in \mathbb{H}$ with $\operatorname{Im} z_0 > R$ and set $w_0 = f(z_0)$. We first note that condition (1.3) can be expressed by

$$\frac{1}{\pi} \iint_{A(t;r,R) \cap \mathbb{H}} \frac{D_{\mu,t}(z) - 1}{|z - t|^2} dx dy \leq \frac{2M}{\pi}$$

for $t \in I$ and $0 < r < R$. Therefore, by (4.3),

$$\operatorname{mod} f(A(t;r,R) \cap \mathbb{H}) \geq \log \frac{R}{r} - \frac{2M}{\pi}.$$

In particular, $\operatorname{mod} f(A(t;r,R) \cap \mathbb{H}) > \pi$ for $0 < r < r_0$, where r_0 is taken so that $\log R/r_0 \geq \pi + 2M/\pi$. Since $f(A(t;r,R) \cap \mathbb{H})$ separates w_0 from $f(t)$ in \mathbb{H} , one has $\operatorname{dist}(f(t), V_2) \leq |f(t) - w_0|$, where V_2 is the unbounded component of $\mathbb{H} \setminus f(A(t;r,R) \cap \mathbb{H})$. We now take an arbitrary point $z \in \mathbb{H}$ with $|z - t| < r_0$ and set $r = |z - t|$. Theorem 2.8 now yields

$$|f(z) - f(t)| \leq C_1 |f(t) - w_0| \exp(-\log \frac{R}{r} + \frac{2M}{\pi}) = C_2 |f(t) - w_0| |z - t|,$$

where $C_2 = C_1 e^{2M/\pi}/R$. Thus we have shown that f is locally Lipschitz continuous on I . \square

Proof of Theorem 1.4. We set

$$\omega(t;r) = \frac{2}{\pi r^2} \iint_{A(t;0,r) \cap \mathbb{H}} D_{\mu,t}(z) dx dy - 1 = \frac{2}{\pi r^2} \iint_{A(t;0,r) \cap \mathbb{H}} (D_{\mu,t}(z) - 1) dx dy.$$

Then (1.5) is equivalent to the condition $\limsup_{r \rightarrow 0+} \omega(t;r) \leq \alpha^{-1} - 1$. Since the convergence is locally uniform in I by assumption, for a compact subset I_0 of I and a given

$0 < \alpha' < \alpha$, we can find a constant $R > 0$ such that $\omega(t; r) \leq 1/\alpha'' - 1$ for $t \in I_0$ and $0 < r \leq R$, where $\alpha'' = (\alpha + \alpha')/2$. On the other hand, we observe that $\omega(t; r) \geq -1$ by definition. In particular, $\omega(t; r)$ is bounded in $0 < r \leq R$ for each $t \in I_0$. We now have the relation

$$\begin{aligned} (Q_\mu(t; r, R) - 1) \log \frac{R}{r} &= \frac{1}{\pi} \iint_{A(t; r, R) \cap \mathbb{H}} \frac{D_{\mu, t}(z) - 1}{|z - t|^2} dx dy \\ &= \frac{\omega(t; R) - \omega(t; r)}{2} + \int_r^R \omega(t; s) \frac{ds}{s} \end{aligned}$$

(see [16, Lemma 3.8]). In particular,

$$(Q_\mu(t; r, R) - 1) \log \frac{R}{r} \leq \frac{1}{2\alpha''} + \int_r^R \left(\frac{1}{\alpha''} - 1 \right) \frac{ds}{s} = \frac{1}{2\alpha''} + \left(\frac{1}{\alpha''} - 1 \right) \log \frac{R}{r},$$

and consequently, $Q_\mu(t; r, R) \leq 1/\alpha'$ for $0 < r < r_0$ and $t \in I_0$ for a sufficiently small $0 < r_0 < R$. By (4.2), we have

$$\text{mod } f(A(t; r, R) \cap \mathbb{H}) \geq \frac{\text{mod}(A(t; r, R) \cap \mathbb{H})}{Q_\mu(t; r, R)} \geq \alpha' \log \frac{R}{r}$$

for $0 < r < r_0$. As in the proof of Theorem 1.2, we have an estimate of the form $|f(z) - f(t)| \leq C|z - t|^{\alpha'}$ for $z \in \mathbb{H}$ $t \in I_0$ with $|z - t| \leq r_0$. \square

Proof of Theorem 1.7. For a fixed $\zeta = e^{i\theta_0} \in \partial\mathbb{D}$, we define a Möbius transformation L by $L(z) = i(\zeta - z)/(\zeta + z)$, $\tilde{L}(z) = i(f(\zeta) - z)/(f(\zeta) + z)$ and let $M = L^{-1}$. Note that L and \tilde{L} map \mathbb{D} conformally onto \mathbb{H} and that $L(\zeta) = 0$, $\tilde{L}(f(\zeta)) = 0$. Let $\hat{\mu}$ be a Beltrami coefficient on \mathbb{H} given by $\hat{\mu} = \mu \circ M \cdot \overline{M'}/M'$. Then $F = \tilde{L} \circ f \circ M$ is a $\hat{\mu}$ -conformal self-homeomorphism of \mathbb{H} . For $w = u + iv = L(z)$, we therefore have the relation

$$\frac{\hat{\mu}(w) du dv}{(w - 0)^2 (1 - |\hat{\mu}(w)|^2)} = \frac{L'(z)^2}{L(z)^2} \frac{\mu(z) dx dy}{(1 - |\mu(z)|^2)}.$$

Since $L'(z)/L(z) = 2\zeta/(z^2 - \zeta^2)$, we see that conditions (i) and (ii) in Theorem 1.7 are equivalent to conditions (1) and (2) in Theorem 1.1 with $t = 0$, respectively. The continuity of f now follows from the argument same as in the proof of Theorem 1.1 and from Proposition 3.1. If we write $e^{i\theta} = M(u)$ for $u \in \mathbb{R}$, we have $ie^{i\theta} d\theta = M(u) du$. In particular, $d\theta/du = 2$ at $u = 0$. Since $F = \tilde{L} \circ f \circ M$ and $\tilde{L}'(0) = 1/(2if(\zeta))$, we have

$$\lim_{u \rightarrow 0} \frac{F(u)}{u} = \frac{1}{if(\zeta)} \lim_{\theta \rightarrow \theta_0} \frac{f(e^{i\theta}) - f(e^{i\theta_0})}{\theta - \theta_0},$$

which implies the differentiability property of f . The continuity of $df(e^{i\theta})/d\theta$ follows from that of Theorem 1.1. \square

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